

Transformations of algebraic Gauss hypergeometric functions

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July 30, 2008

Abstract

A celebrated theorem of Klein implies that any hypergeometric differential equation with algebraic solutions is a pull-back of one of the few standard hypergeometric equations with algebraic solutions. The most interesting cases are hypergeometric equations with tetrahedral, octahedral or icosahedral monodromy groups. We give an algorithm for computing Klein's pull-back coverings in these cases, based on certain explicit expressions (Darboux evaluations) of algebraic hypergeometric functions. The explicit expressions can be computed from a data base (covering the Schwarz table) and using contiguous relations. Klein's pull-back transformations also induce algebraic transformations between hypergeometric solutions and a standard hypergeometric function with the same finite monodromy group.

1 Introduction

Algebraic Gauss hypergeometric functions is a very classical object of research in mathematics. They were studied by many authors [Sch72, Fuc75, Bri77, Kle78, Pep81, Bou98, Kat72, BD79, SU93, vdPU98]. Recall that Gauss hypergeometric function ${}_2F_1\left(\begin{smallmatrix} A, B \\ C \end{smallmatrix} \middle| Z\right)$ satisfies the *hypergeometric differential equation*:

$$Z(1-Z) \frac{d^2 Y(Z)}{dZ^2} + (C - (A+B+1)Z) \frac{dY(Z)}{dZ} - AB Y(Z) = 0. \quad (1)$$

This is a Fuchsian equation with three regular singular points $Z = 0, 1$ and ∞ on \mathbf{P}^1 . The local exponent differences at these points are (up to a sign) $1 - C$, $C - A - B$ and $A - B$ respectively. We denote equation (1) by $H(1 - C, C - A - B, A - B)$. With this notation, equation the $H(e_0, e_1, e_\infty)$ can be written in the following form:

$$\frac{d^2 Y(Z)}{dZ^2} + \left(\frac{1 - e_0}{Z} + \frac{1 - e_1}{Z - 1} \right) \frac{dY(Z)}{dZ} + \frac{(1 - e_0 - e_1)^2 - e_\infty^2}{4Z(Z - 1)} Y(Z) = 0. \quad (2)$$

If we permute the parameters e_0, e_1, e_∞ or multiply some of them by -1 , we get hypergeometric equations related by well-known fractional-linear transformations, see [AAR99]. In

general, there are 24 hypergeometric equations related in this way, and they share the same (up to fractional-linear transformations of the argument and radical factors) 24 hypergeometric Kummer's solutions.

As is known (see [Beu02] for example), a Fuchsian equation only has algebraic solutions if and only if its *monodromy group* is finite. The following hypergeometric equations (and their fractional-linear transformations just mentioned) have this property and are called *standard hypergeometric equations* with algebraic solutions:

- $H(1, 1/n, 1/n)$, where n is a positive integer. The hypergeometric equation degenerates to a Fuchsian equation with two singular points. Its monodromy group is the cyclic group with n elements.
- $H(1/2, 1/2, 1/n)$, where n is an integer, $n \geq 2$. The monodromy group of this equation is the dihedral group with $2n$ elements.
- $H(1/2, 1/3, 1/3)$. The monodromy group is the tetrahedral group, isomorphic to A_4 .
- $H(1/2, 1/3, 1/4)$. The monodromy group is the octahedral group, isomorphic to S_4 .
- $H(1/2, 1/3, 1/5)$. The monodromy group is the icosahedral group, isomorphic to A_5 .

The celebrated theorem of Klein [Kle77] states that if a second order linear homogeneous equation only has algebraic solutions, then it is a *pull-back transformation* of a standard hypergeometric equation from the list above. If the Fuchsian equation has coefficients in $\mathbf{C}(X)$, such a pull-back transformation changes the variable Z in (2) to a rational function $\varphi(X)$. In geometric terms, we have a finite covering $\varphi : \mathbf{P}_X^1 \rightarrow \mathbf{P}_Z^1$ between two projective lines, and we *pull-back* the standard hypergeometric equation from \mathbf{P}_Z^1 onto \mathbf{P}_X^1 .

Particularly, the theorem of Klein implies that if a hypergeometric equation H_1 only has algebraic solutions, then it is a pull-back transformation of a standard hypergeometric equation H_0 with algebraic solutions. The proof of Klein implies that the monodromy group of H_1 is either cyclic, or dihedral, or A_4 , S_4 or A_5 , and that H_0 can be chosen to be a standard hypergeometric equation with the same monodromy group. The main purpose of this paper is to give an algorithm for computing the pull-back transformations implied by the theorem of Klein. We consider here the most interesting cases of hypergeometric equations with tetrahedral, octahedral or icosahedral monodromy groups. Hypergeometric equations with dihedral monodromy groups are considered in [Vid08].

Hypergeometric equations with algebraic solutions were first classified by Schwarz in [Sch72]. Disregarding hypergeometric equations with a cyclic monodromy group, Schwarz gave a list of fifteen types of these hypergeometric equations. One type consists of hypergeometric equations with a dihedral monodromy group. The other types are represented by the following hypergeometric equations:

- $H(1/2, 1/3, 1/3)$, $H(1/3, 1/3, 2/3)$. The monodromy group is the tetrahedral group.
- $H(1/2, 1/3, 1/4)$, $H(2/3, 1/4, 1/4)$. The monodromy group is the octahedral group.

- $H(1/2, 1/3, 1/5), H(1/2, 1/3, 2/5), H(1/2, 1/5, 2/5), H(1/3, 1/3, 2/5), H(1/3, 2/3, 1/5),$
 $H(2/3, 1/5, 1/5), H(1/3, 2/5, 3/5), H(1/3, 1/5, 3/5), H(1/5, 1/5, 4/5), H(2/5, 2/5, 2/5).$

The monodromy group is the icosahedral group.

We refer to Schwarz type of hypergeometric equations with algebraic solutions by the triple of the parameters e_0, e_1, e_∞ of these representative equations. (Usually, Schwarz type is a roman numeral from I to XV.) Hypergeometric equations of the same Schwarz type are characterized by the property that their hypergeometric solutions are *contiguous* to hypergeometric solutions of the representative hypergeometric equation of that type. For example, the parameters e_0, e_1, e_∞ of hypergeometric equations of the Schwarz type $(1/3, 1/3, 2/3)$ can be characterized as follows: they are rational numbers, their denominators are equal to 3, and the sum of their numerators is even. See [Vid05a, Subsection 5.6] for more details.

Recently, M. van Hoiej, J.-A. Weil and M. Berkenbosch [BvHW03], [vHW05] developed an algorithm for computing Klein's pull-back transformations for second order linear differential equations, based on computation of monodromy (or differential Galois group) semi-invariants, via explicit solution of suitable symmetric tensor powers of those differential equations. extension of Klein's theorem to third order linear differential equations equations with a finite monodromy group is possible [Ber04].

The algorithm in this paper is devised for hypergeometric equations with algebraic solutions. Instead of finding semi-invariants by solving symmetric tensor power equations, we propose to use contiguous relations and a data base of explicit expressions for simplest hypergeometric solutions. This approach should be more effective when local exponent differences of hypergeometric equations are large, since solving symmetric power equations apparently involves a linear algebra problem whose size is proportional to the local exponent differences, while deriving a contiguous relation can be of logarithmic complexity [Vid03]. On the other hand, direct transformation of explicit expressions by Gröbner basis methods appears border-line cumbersome for differential equations with the icosahedral monodromy group. But these transformations can be apparently streamlined in each of the icosahedral Schwarz cases using a normal series of the solvable icosahedral group, as elimination degrees are universally bounded. This approach simplifies greatly for algebraic dihedral hypergeometric functions, as demonstrated in [Vid08].

The input of our algorithm is a hypergeometric equation with a tetrahedral, octahedral or icosahedral monodromy group. The output is a finite covering $\varphi : \mathbf{P}^1 \rightarrow \mathbf{P}^1$ which pull-backs the corresponding standard hypergeometric equation to the given one. With a little more work one may start with an algebraic hypergeometric Gauss function, and produce an algebraic transformation which relates the given hypergeometric function to hypergeometric solutions of the corresponding standard hypergeometric equation. The algorithm is based on the following three ingredients:

- *Pull-back transformation of a hypergeometric equation to a Fuchsian equation (not necessarily on \mathbf{P}^1) with a cyclic monodromy group.* Geometrically, we pull-back the hypergeometric equation onto its *Darboux curve* by a *Darboux covering*; see Section 2. The algebraic hypergeometric solutions are transformed to *radical functions* (i.e., products of powers of rational functions on the Darboux curve). This transformation

allows us to write algebraic Gauss functions in a convenient explicit form. General theory and computations of Darboux coverings and evaluations is presented in [Vid05a].

- *Contiguous relations of Gauss hypergeometric functions.* With a pre-computed data basis of Darboux coverings and explicit Darboux evaluations for simplest algebraic hypergeometric functions of each Schwarz type at hand, we use contiguous relations to compute any algebraic algebraic function in that explicit form. See [Vid05a, Subsection 5.6].
- *Schwarz map.* As a function, this is a quotient of two solutions of the same hypergeometric equation. If s_0 is a Schwarz map for a standard hypergeometric equations with algebraic solutions, then s_0^{-1} is a finite Galois covering $\mathbf{P}^1 \rightarrow \mathbf{P}^1$ on which the monodromy group acts as the Galois group. Schwarz map plays an essential role in the proof of Klein [Kle77]; see Section 2. Klein's pull-back covering is identical to a Schwarz map of the given hypergeometric equation composed with the rational function s_0^{-1} of the corresponding standard equation. We make this construction very explicit using the previous two ingredients.

Our algorithm is presented in Section 3. Sections 4 and 5 demonstrate the algorithm by several examples. In Section 6 we present the mentioned data base of Darboux coverings and explicit hypergeometric evaluations.

2 Some preliminaries

Here we bring forward the most important notions for our algorithm. A more comprehensive presentation of related theory (with further references), and computation of the data base in Section 6 are given in the supplementing paper [Vid05a].

First we briefly sketch the Klein's proof of his theorem in [Kle77] and [Kle84, Section V.3]. The theorem states that if the differential equation $Y'' + pY' + q = 0$ with $p, q \in \mathbf{C}(X)$ only has algebraic solutions, it is a pull-back transformation of one of the standard hypergeometric equations. Let $s \in \mathbf{C}(X)$ denote a quotient of two distinct solutions of this equations. Then s satisfies the equation

$$\frac{s'''}{s'} - \frac{3}{2} \left(\frac{s''}{s'} \right)^2 = 2q - p' - \frac{1}{2} p^2. \quad (3)$$

Incidentally, the expression on the left-hand side is the *Schwarzian derivative* $\{s, X\}$. Solutions of this equation are precisely the fractional-linear transformations $(as+b)/(cs+d)$ of s . Therefore, analytic continuation of s along a closed loop (away from singularities of the given equation) changes s by a fractional-linear transformation. All closed paths with a common fixed point give a group of fractional-linear transformations that acts on the branches of s . This is the *projective monodromy group* of the equation; see [Vid05a, Section 5.4] for example. Since s is an algebraic function, this group must be finite. Finite groups of fractional-linear transformations are well-known: such a group is either cyclic, or dihedral, or A_4 , S_4 or A_5 . Functions in $\mathbf{C}(s)$ that are invariant under the transformation group are rational functions of X . When a suitable invariant function is equal to X , the original equation $Y'' + pY' + q = 0$

is a standard hypergeometric equation with the same projective monodromy group. In general, the same invariant function (as a rational function of X) gives the required pull-back transformation from the same standard hypergeometric equation.

If the equation $Y'' + pY' + q = 0$ is a hypergeometric equation, we interpret the construction of Klein as follows. The quotient s of two solutions is a *Schwarz map* for the hypergeometric equation, see [Vid05a, Section 5.5] for example. If s_0 is the quotient of two corresponding solutions of the standard hypergeometric equation with the same monodromy group, then s_0^{-1} is the “suitable” invariant function that cancels the monodromy of s . Klein’s pull-back covering is the composition $s_0^{-1} \circ s$.

A way to obtain Klein’s pull-back covering is to compute the quotient s of two convenient hypergeometric solutions as a root of an algebraic equation, and then use elimination techniques to compute the composition $s_0^{-1} \circ s$ as a rational function of X . We particularly propose to use a convenient expression for the function s by invoking *Darboux curves* and *Darboux coverings*, so to make computation of $\psi(s)$ algebraically straightforward. Specifically, we propose to pull-back the given differential equation to its Darboux curve, and express its hypergeometric solutions and the quotient s of two solutions as radical functions on the Darboux curve. This is a very convenient and explicit expression for the function s .

In the remainder of this Section we present and comment our working definition of Darboux curves and Darboux coverings and sketch their application to our algorithm. We refer to [Vid05a, Section 3] for a thorough discussion of Darboux curves and Darboux coverings, and to [Vid05a, Section 5.3] for our specifications of pull-back transformations.

Definition 2.1 Let E denote a linear homogeneous differential equation with coefficients in $\mathbf{C}(X)$. Let \mathbf{D} be an algebraic curve, and let $\mathbf{C}(\mathbf{D})$ be the field of rational functions on \mathbf{D} . Suppose that $\varphi : \mathbf{D} \rightarrow \mathbf{P}^1$ is a finite covering. We say that \mathbf{D} is a *Darboux curve* for E , and that φ is a *Darboux covering* for E , if a pull-back transformation of E onto D with respect to φ has a solution Y which satisfies $Y' = uY$ for some $u \in \mathbf{C}(\mathbf{D})$, and the algebraic degree of u over $\mathbf{C}(X)$ is equal to the degree of φ .

In general, a pull-back transformation of a differential equation allows projective transformation $y(x) \mapsto \theta(x)y(x)$ of solutions by a radical function $\theta(x)$. A general pull-back transformation has the form

$$z \mapsto \varphi(x), \quad y(z) \mapsto \theta(x)y(\varphi(x)). \quad (4)$$

When speaking about Darboux curves, it is more convenient to refer to the *differential Galois group* of a differential equation rather than to the monodromy group. In the case of hypergeometric equations with algebraic solutions only, these two groups coincide. Particularly, the Piccard-Vessiot extension (where the solutions “live”) is an algebraic Galois extension, it is generated over $\mathbf{C}(X)$ by a solution of the hypergeometric equation, and its usual Galois group is the same monodromy group. For basic notions of differential Galois theory and further references (in particular, to proofs of mentioned facts) we refer to [Vid05a, Section 5.2].

Consider a hypergeometric equation H_1 with algebraic solutions only. Let $K \supset \mathbf{C}(X)$ be its Piccard-Vessiot extension, and let G denote its differential Galois group. If \mathbf{D} is a

Darboux curve for H_1 , then its function field $\mathbf{C}(\mathbf{D})$ is an intermediate field: $K \supset \mathbf{C}(\mathbf{D}) \supset \mathbf{C}(X)$. Besides, $K \cong \mathbf{C}(X, Y)$ and $\mathbf{C}(\mathbf{D}) \cong \mathbf{C}(X, u)$ for $u = Y'/Y$ and for a suitable solution Y of H_1 . The extension $K \supset \mathbf{C}(\mathbf{D})$ must be a finite subgroup of the multiplicative group $\mathrm{GL}(1, \mathbf{C})$, so it must be a finite cyclic group. Conversely, each cyclic subgroup of G defines a Darboux curve. In our algorithm we use Darboux curves which correspond to cyclic subgroups of maximal order. These Darboux curves have the smallest genus, and the corresponding Darboux coverings have minimal degree; see [Vid05a, Table 1].

From now on, let X denote the indeterminate of a hypergeometric equation with algebraic solutions, and let Z denote the indeterminate of a standard hypergeometric equation with the same differential Galois group G . By [Vid05a, Lemma 3.3(i)], all Darboux curves for the standard hypergeometric equation are rational; we denote a rational parameter of such a Darboux curve by z . In the current context, let H_0 denote the standard hypergeometric equation for H_1 , and consider a Darboux covering $\varphi_0 : \mathbf{P}_z^1 \rightarrow \mathbf{P}_Z^1$ for H_0 of minimal degree. Let $\psi : \mathbf{P}_X^1 \rightarrow \mathbf{P}_Z^1$ be the Klein covering which pull-backs H_1 to H_0 , and let $\varphi_1 : \mathbf{D} \rightarrow \mathbf{P}_X^1$ be the Darboux covering for H_1 of minimal degree. By [Vid05a, Lemma 3.5], the Darboux curve \mathbf{D} for H_1 is the fiber product of $\varphi_0 : \mathbf{P}_z^1 \rightarrow \mathbf{P}_Z^1$ and $\psi : \mathbf{P}_X^1 \rightarrow \mathbf{P}_Z^1$, and φ_1 is the projection onto \mathbf{P}_X^1 . We have the following diagram:

$$\begin{array}{ccccc}
 & \mathbf{P}_s^1 & & \mathbf{D} & \\
 & \searrow \gamma & & \searrow \varphi_1 & \\
 & & \mathbf{P}_z^1 & & \mathbf{P}_X^1 \\
 & & \searrow \varphi_0 & & \searrow \psi \\
 & & & \mathbf{P}_Z^1 &
 \end{array} \tag{5}$$

Here the composition $\varphi_0 \circ \gamma : \mathbf{P}_s^1 \rightarrow \mathbf{P}_Z^1$ is the inverse Schwarz map s_0^{-1} for H_0 , and formally, a Darboux covering for H_0 of maximal degree. Besides, $\mathbf{C}(s) \supset \mathbf{C}(Z)$ is the Piccard-Vessiot extension for the standard hypergeometric equation. By [Vid05a, Lemma 3.3], the map $\gamma : \mathbf{P}_s^1 \rightarrow \mathbf{P}_z^1$ is a Galois covering, and its Galois group is a maximal cyclic subgroup of G .

The role of Darboux coverings in our main Algorithm 3.1 can be viewed in two ways. On one hand, we can identify the hypergeometric solutions of H_1 and H_0 up to unknown scalar factor; see our examples in Section 4 below. The scalar factor can be determined by evaluating the hypergeometric functions on some corresponding points on the Darboux curves $\mathbf{D} \rightarrow \mathbf{P}_z^1$. Once we choose a point P on \mathbf{P}_z^1 , there are (a priori) finitely many choices on \mathbf{D} in the same fiber of $\varphi_1 \circ \psi$ above $\varphi_0(P)$ for this identification. A further detailed analysis of all Schwartz cases may define unique identifications here. To identify a right pair of corresponding points, one can check the possibilities using a pair of compatible automorphisms of the two Darboux coverings.

Let f_1, f_2 be two hypergeometric solutions of H_1 identified in this way with two hypergeometric solutions g_1, g_2 (respectively) of H_0 . The pull-backs of f_1, g_1 (and of f_2, g_2) onto the respective Darboux curves $\mathbf{P}_z^1, \mathbf{D}$ can be identified, once the scalar factor is determined. It is important in computations that those pull-back functions are radical functions on the

Darboux curves, so they have convenient explicit expressions. The right identification of the quotients f_1/f_2 and g_1/g_2 determines Klein's pull-back covering ψ . This identification of f_1/f_2 and g_1/g_2 on the Darboux curve \mathbf{D} , and explicit expressions for φ_1 , φ_0 give enough equations to determine an algebraic relation between X and Z . Klein's theorem ensures that factorization of this relation gives the expression $Z = \psi(X)$ for Klein's pull-back covering.

Other interpretation is conceptually more straightforward. Note that f_1/f_2 and $s_0 = g_1/g_2$ are Schwarz maps for H_1 and H_0 respectively. The construction of Klein implies that $s_0^{-1}(f_1/f_2)$ is precisely Klein's covering, once the two quotients are rightly identified. We use Darboux curves here to have a convenient explicit expression for the quotient f_1/f_2 as a function on \mathbf{D} . We can make standardized identifications and use the most handy Galois invariant functions s_0^{-1} for the tetrahedral, octahedral and icosahedral cases. Klein's theorem ensures that in our expression of $s_0^{-1}(f_1/f_2)$ we can eliminate the variables of $\mathbf{C}(\mathbf{D})$ and get the required function $\psi(X)$. This interpretation gives This is a convenient conceptual interpretation for our Algorithm 3.1.

The relation of Schwarz maps and Klein's pull-back covering is surely widely known. Our use of Darboux coverings for explicit computations is perhaps new. It is interesting to note that in [OY04] Ochiai and Yoshida compare the Schwarz maps s and s_0 by considering the dual composition $s \circ s_0^{-1}$. They consider only hypergeometric equations of standard Schwarz types, so their hypergeometric functions are contiguous to hypergeometric solutions of the standard equations $H(1/2, 1/3, 1/3)$, $H(1/2, 1/3, 1/4)$, $H(1/2, 1/3, 1/5)$. Then the composition $s \circ s_0^{-1}$ is a finite covering $\mathbf{P}^1 \rightarrow \mathbf{P}^1$ as well. The finite coverings $s_0^{-1} \circ s$, $s \circ s_0^{-1}$ are related by fiber product construction via two copies of s_0^{-1} :

$$\begin{array}{ccc}
 \mathbf{P}_{s_0}^1 & \xrightarrow{s \circ s_0^{-1}} & \mathbf{P}_s^1 \\
 \searrow s_0^{-1} & & \nearrow s^{-1} \\
 & \mathbf{P}_x^1 & \xrightarrow{s_0^{-1} \circ s} \mathbf{P}_z^1 \\
 & \nearrow s_0^{-1} & \searrow s^{-1}
 \end{array}$$

Here on \mathbf{P}_x^1 we consider a hypergeometric equation with algebraic solutions, on \mathbf{P}_z^1 — the corresponding standard equation, and on $\mathbf{P}_{s_0}^1$ — both hypergeometric equations. The covering $s \circ s_0^{-1}$, as a rational function of s_0 , can be conveniently computed in a similar way, using the same explicit expression of s as a function on the Darboux curve.

3 Computation of Klein's pull-back coverings

Here we present an algorithm for computing Klein's coverings. As mentioned in the introduction, it is based on explicit evaluations of algebraic Gauss hypergeometric functions in Section 6 (which use Darboux coverings); contiguous relations (by which we get similar explicit expressions for any hypergeometric functions); and identification of Schwarz maps (to get an extra algebraic relation between parameters on different curves).

Algorithm 3.1 *Input:* A hypergeometric equation H_1 with tetrahedral, octahedral or icosahedral monodromy group.

Output: A finite covering $\psi : \mathbf{P}^1 \rightarrow \mathbf{P}^1$, which pull-backs a standard hypergeometric equation H_0 with the same monodromy group to H_1 .

Step 0. As in the previous section, we denote the variables of the differential equations H_0 and H_1 by Z and X respectively. The local exponent differences of H_0 are $1/2, 1/3, 1/m$, where $m = 3, 4$ or 5 for the tetrahedral, octahedral or icosahedral monodromy groups, respectively. We assign the local exponent differences $1/2, 1/3, 1/m$ to the points $Z = 1, Z = \infty$ and $Z = 0$, respectively.

Step 1. We assume that the point $X = 0$ of the given equation lies above the point $Z = 0$ under Klein's pull-back covering. The points $X = 1$ and $X = \infty$ must be assigned so that they would lie above the Z -points with the same denominators of local exponent differences. Particularly, if the given equation has a half-integer local exponent, we assume that the point $X = 1$ has that local exponent, and that it lies above $Z = 1$.

Step 2. We use the Darboux coverings and evaluations explicitly presented in Section 6. Let z denote a rational parameter for the Darboux curve for the standard equation H_0 . We denote coordinate functions of the Darboux curve for H_1 by x and ξ . (If the Darboux curve is rational, then $\xi = 0$ and we use the variable x only.) Let $\phi_0(z), \phi_1(x, \xi)$ denote the Darboux coverings for H_0, H_1 respectively. We have the algebraic relations $Z = \phi_0(z), X = \phi_1(x, \xi)$ and an algebraic relation between x, ξ (which possibly $\xi = 0$).

Step 3. Consider the following solutions of H_0 :

$$F_1 = {}_2F_1\left(\begin{matrix} \frac{1}{12} - \frac{1}{2m}, \frac{5}{12} - \frac{1}{2m} \\ 1 - \frac{1}{m} \end{matrix} \middle| Z\right), \quad F_2 = Z^{1/m} {}_2F_1\left(\begin{matrix} \frac{1}{12} + \frac{1}{2m}, \frac{5}{12} + \frac{1}{2m} \\ 1 + \frac{1}{m} \end{matrix} \middle| Z\right) \quad (6)$$

Let $f_0(Z)$ be the quotient of these two functions (a ‘‘Schwarz map’’ for H_0). We evaluate $f_0(\phi_0(z))$ using suitable hypergeometric identities from the data base in Section 6. Up to a constant multiple, the Schwarz maps are equal to $z^{-1/m}$.

Step 4. Consider two hypergeometric solutions of H_1 ,

$$G_1 = {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix} \middle| X\right), \quad G_2 = X^{1-c} {}_2F_1\left(\begin{matrix} 1+a-c, 1+b-c \\ 2-c \end{matrix} \middle| X\right), \quad (7)$$

corresponding to the functions in (6). The local exponent differences $1-c, c-a-b$ and (say) $a-b$ are the positive local exponent differences of the given equation. In particular, $c < 1$ and $c-a-b > 0$. Besides, c is a rational number with the denominator m . The functions F_1 and G_1 are identified under Klein's pull-back transformation up to a power factor. The functions F_2 and G_2 are identified up to the same power factor and a constant multiple.

Step 5. In the data base of Section 6 there are four evaluations of hypergeometric functions with the Schwarz type of H_1 . Possibly up to Euler's transformation [AAR99, (2.2.7)] two of them are contiguous to G_1 , and the other two are contiguous to G_2 . We find a contiguous relations between G_1 (or its Euler's transformation) and the former two data base functions, and a contiguous relation between G_2 (or its Euler's transformation) and the latter two data base functions. We evaluate the functions using the contiguous relations and the data base

evaluations. The data base evaluations introduce radical expressions in x, ξ . Coefficients of the contiguous relations are rational functions of X ; they should not be evaluated in terms of x, ξ . The resulting expressions of the functions G_1 and G_2 have the following form: a fractional power expression in x and possibly ξ (which can be standardized for each Schwarz type), times a rational function in X, x and possibly ξ (of bounded degree in x, ξ).

Step 6. Let $f_1(X)$ denote the Schwarz map G_1/G_2 for H_1 . We consider $f_1(X)$ explicitly evaluated like the functions G_1, G_2 in the previous step: a standardized fractional power function in x, ξ and a rational function in X, x, ξ . The Schwarz maps $f_0(Z)$ and $f_1(X)$ are identified up to a constant multiple w by Klein's pull-back covering. The identification gives a rational relation $z = w \Phi(X, x, \xi)$. To find the constant w , we evaluate this identity at the following pairs of z - and (x, ξ) -points, lying above $Z = 0$ and $X = 0$, respectively. In the icosahedral cases, we have two possibilities for the z -point; the right possibility is determined by checking compatibility of the given automorphisms of the z - and (x, ξ) Darboux coverings.

- For the tetrahedral type $(1/2, 1/3, 1/3)$: evaluation at $x = -4$, or $z = -4$.
- For the tetrahedral type $(1/3, 1/3, 2/3)$: evaluation at $x = -2$.
- For the octahedral type $(1/2, 1/3, 1/4)$: evaluation at $x = 1$, or $z = 1$.
- For the octahedral type $(2/3, 1/4, 1/4)$: evaluation at $x = -1$.
- For the icosahedral types $(1/2, 1/3, 1/5)$ and $(1/2, 1/3, 2/5)$:
evaluation at $x = (11 + 5\sqrt{5})/2$, or $z = (11 \pm 5\sqrt{5})/2$;
the automorphism $x \mapsto -1/x$, or $z \mapsto -1/z$.
- For the icosahedral type $(1/2, 1/5, 2/5)$: evaluation at $x = (1 + \sqrt{5})/2$;
the automorphism $x \mapsto -1/x$.
- For the icosahedral types $(1/3, 1/3, 2/5)$ and $(1/3, 2/3, 1/5)$:
evaluation at $(x, \xi) = ((11 + 5\sqrt{5})/6, 0)$; the automorphism $x \mapsto -1/9x$, $\xi \mapsto \xi/9x^2$.
- For the icosahedral types $(2/3, 1/5, 1/5)$ and $(1/3, 2/5, 3/5)$:
evaluation at $(x, \xi) = ((-3 + 7/\sqrt{5})/2, 7 - 3\sqrt{5})$;
the automorphism $x \mapsto -1/5x$, $\xi \mapsto \xi/5x^2$.
- For the icosahedral type $(1/3, 1/5, 3/5)$:
evaluation at $(x, \xi) = ((-3 + \sqrt{5})/8, (-5 + 3\sqrt{5})/8)$;
the automorphism $x \mapsto 1/16x$, $\xi \mapsto -\xi/16x^2$.
- For the icosahedral types $(1/5, 1/5, 4/5)$ and $(2/5, 2/5, 2/5)$:
evaluation at $(x, \xi) = ((1 + \sqrt{5})/2, 0)$; the automorphism $x \mapsto -1/x$, $\xi \mapsto \xi/x^2$.

Step 7. We have three algebraic equations produced by Step 2, and the equation $z = w \Phi(X, x, \xi)$ produced by Step 6. The variables are Z, X, z, x and ξ . The equations have bounded degree in z, x, ξ . Using Gröbner bases techniques (or resultants) we eliminate z, x and (if relevant) ξ , and get a polynomial relation between X and Z . Klein's theorem ensures

this polynomial relation has a factor linear in Z , giving the solution $Z = \Phi(X)$. \square

This algorithm is demonstrated by examples in Sections 4 and Section 5. Here we give several comments to some steps.

Let (e_0, e_1, e_∞) denote the positive local exponent differences of the given equation H_1 . The degree d of Klein's pull-back covering can be computed using part 2 of [Vid04, Lemma 2.5]:

$$d = \frac{e_0 + e_1 + e_\infty - 1}{\frac{1}{2} + \frac{1}{3} + \frac{1}{m} - 1} = \frac{6m}{6 - m}(e_0 + e_1 + e_\infty - 1). \quad (8)$$

This formula has an interpretation as the area ratio of the Schwarz triangles for the given and standard hypergeometric equations.

Steps 0 and 3 are the same for input differential equations with same monodromy group. Steps 1, 2 and probably 6 can be set up to work in the same way for the input of the same Schwarz type.

In Step 3, let $s = z^{-1/m}$ denote the Schwarz maps for a standard tetrahedral, octahedral and icosahedral equation. The inverse Schwarz map is a rational function $S_m(s)$ of s (with $m = 3, 4, 5$). We recognize classical invariants of the Galois groups in these expressions:

$$\begin{aligned} S_3(s) &= \frac{s^3 (s^3 + 4)^3}{4(2s^3 - 1)}, & S_4(s) &= \frac{108 s^4 (s^4 - 1)^4}{(s^8 + 14s^4 + 1)^3}, \\ S_5(s) &= \frac{1728 s^5 (s^{10} - 11s^5 - 1)^5}{(s^{20} + 228s^{15} + 494s^{10} - 228s^5 + 1)^3}. \end{aligned}$$

In Step 4, the parameters a, b, c have the following expressions in terms of the local exponent differences:

$$a = \frac{1 - e_0 - e_1 - e_\infty}{2}, \quad a = \frac{1 - e_0 - e_1 + e_\infty}{2}, \quad c = 1 - e_0. \quad (9)$$

Euler's fractional-linear transformation [AAR99, (2.2.7)] in Step 5 reads

$${}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix} \middle| z\right) = (1 - z)^{c-a-b} {}_2F_1\left(\begin{matrix} c - a, c - b \\ c \end{matrix} \middle| z\right). \quad (10)$$

The function G_1 (or its Euler's transformation) might be contiguous to the last two of the four evaluations of the relevant Schwarz type in the data base, while the function G_2 (or its Euler's transformation) would be contiguous to the first two data base evaluations; this is demonstrated in Example 4.2. There are (at most) 4 different contiguity orbits for G_1 (and G_2) for each Schwarz type; for each of those orbits, the constant w in the next step can probably be determined by a fixed evaluation of the relation $z = w\Psi(X, x, \xi)$ even in icosahedral cases.

Note that each pair of contiguous evaluations in the data base is presented with the same fractional-power part, in the cases of genus 1 Darboux curves as well. This allows the algorithm to keep the same fractional-power part in all contiguous evaluations, keeping the same pattern of (x, ξ) variables for Step 7 in each Schwarz case.

In Step 6, we evaluate the relation $z = w\Psi(X, x, \xi)$ at a (x, ξ) -point above $X = 0$ (but not $x = 0$) and at a z -point above $Z = 0$ (but not $z = 0$). Analysis of diagram (5) for each Schwarz type shows that the (x, ξ) point lies above the z -point, except that we may have a choice of two z -values in the icosahedral cases. We check the two z -values by the given automorphisms of Darboux curves. On the other hand, just compatibility of the given automorphisms determines the constant multiple w up to a sign. Evaluation at $X = 1$, $Z = 1$ using Gauss formula [AAR99, Theorem 2.2.2] can be used, but the $X = 1$ value may need to be twisted to a correct branch (which probably depends only on the contiguity orbit of G_1); see Example 4.1.

In Step 7, particular elimination tricks (while using a modern computer algebra system) are essentially needed only in icosahedral cases. Examples suggest that elimination using Gröbner bases or resultants returns an integer power of $Z - \Phi(X)$, with cleared denominator. The icosahedral relations have degree 12 (or a little higher) in the variable x , which is hard to eliminate directly, especially together with $\xi \neq 0$ on genus 1 Darboux curves. The respective automorphism symmetry of Step 6 can be used to half the degree in x . The new variable would look like $\alpha = x - 1/x$ or so; see Example 5.1. As the relation $z = w\Psi(X, x, \xi)$ and the output have highly factorizable forms, elimination can be done fast for each factor, and the resulting factors in X can be combined and then normalized using evaluation at $X = 1$ or differentiation (to find ramification points that must be above $Z = 1$); see Example 5.1. Algebraic computations can probably be streamlined for each Schwarz type using a normal series of the solvable monodromy group and respective resolvents, with simplification at each step to a variable of higher invariance.

4 Tetrahedral examples

Here we demonstrate Algorithm 3.1 on several hypergeometric equations with the tetrahedral monodromy group. The standard hypergeometric equation for this monodromy group has the local exponent differences $(1/2, 1/3, 1/3)$. Following Step 0, we assign the local exponent difference $1/3$ to the points $Z = 0$ and $Z = \infty$. The local exponent difference $1/2$ is assigned to the point $Z = 1$. The Darboux covering $Z = \phi_0(z)$ in Step 2 is

$$Z = \frac{z(z+4)^3}{4(2z-1)^3}. \quad (11)$$

The Schwarz map in Step 3 is

$$f_0(Z) = {}_2F_1\left(\frac{1}{4}, -\frac{1}{12} \middle| \frac{2}{3} \right) Z \bigg/ Z^{1/3} {}_2F_1\left(\frac{1}{4}, \frac{7}{12} \middle| \frac{4}{3} \right) Z = \left(-\frac{1}{16z}\right)^{1/3}, \quad (12)$$

evaluated using (11) and (42), (44).

The first three examples are of the same Schwarz type $(1/2, 1/3, 1/3)$. In accordance with Step 1, the points $X = 0$, $X = 1$, $X = \infty$ are assumed to lie above the points $Z = 0$, $Z = 1$, $Z = \infty$, respectively. The Darboux curve for those hypergeometric equations is rational,

hence $\xi = 0$. The Darboux covering $X = \phi_1(x)$ in Step 2 mimics (11):

$$X = \frac{z(z+4)^3}{4(2z-1)^3}. \quad (13)$$

Example 4.1 We compute Klein's pull-back covering for hypergeometric equations with the local exponent differences $(1/2, 1/3, 2/3)$. The degree is expected to be 3.

The Schwartz map in Step 4 is

$${}_2F_1\left(-\frac{1}{4}, \frac{5}{12} \middle| X\right) \bigg/ X^{1/3} {}_2F_1\left(\frac{3}{4}, \frac{1}{12} \middle| X\right), \quad (14)$$

The two ${}_2F_1$ functions are not contiguous to the data base functions (42)–(45), but by Euler's transformation (10) they are equal, respectively, to

$$\sqrt{1-X} {}_2F_1\left(\frac{1}{4}, \frac{11}{12} \middle| X\right), \quad \sqrt{1-X} {}_2F_1\left(\frac{5}{4}, \frac{7}{12} \middle| X\right).$$

We have the following contiguous relations for these functions, with $(a, b, c) = (1/4, -1/12, 2/3)$ or $(1/4, 7/12, 4/3)$, respectively:

$$bc(1-X) {}_2F_1\left(\frac{a, b+1}{c} \middle| X\right) = c((a-b)X+b) {}_2F_1\left(\frac{a, b}{c} \middle| X\right) + a(b-c)X {}_2F_1\left(\frac{a+1, b}{c+1} \middle| X\right), \quad (15)$$

$$a(1-X) {}_2F_1\left(\frac{a+1, b}{c} \middle| X\right) = (a+1-c) {}_2F_1\left(\frac{a, b}{c} \middle| X\right) + (c-1) {}_2F_1\left(\frac{a, b-1}{c-1} \middle| X\right). \quad (16)$$

Using data base formulas (42)–(45), Step 5 returns

$$\begin{aligned} {}_2F_1\left(-\frac{1}{4}, \frac{5}{12} \middle| \frac{x(x+4)^3}{4(2x-1)^3}\right) &= (1+x)(1-2x)^{-3/4}, \\ {}_2F_1\left(\frac{3}{4}, \frac{1}{12} \middle| \frac{x(x+4)^3}{4(2x-1)^3}\right) &= \frac{1}{1+\frac{1}{4}x}(1-2x)^{1/4}. \end{aligned}$$

In terms of x , the Schwarz map $f_1(X)$ in (14) is a constant times $x^{-1/3}(x+1)$. This must be proportional to (12). Hence $z = wx/(x+1)^3$, where the constant $w = -27$ is obtained by evaluation $(x, y) = (-4, -4)$. In the final step, we eliminate x, z using (13) and (11) by whatever straightforward method, and obtain the result $Z = 27X/(4X-1)^3$. One can try and note that computations become ugly when w is taken wrongly.

The cubic transformation $Z = 27X/(4X-1)^3$ is the same as in a classical cubic transformation of Gauss hypergeometric functions.

Direct comparison of Schwarz maps (12) and (19) by their evaluations at, respectively, $Z = 1$ and $X = 1$, respectively, does not a right identification. Evidently, as X varies from 0 to 1, Z leaves the convergence disk, goes through the infinity, and approaches 1 on other

branch. The explicit evaluations at $Z = 1$ and $X = 1$ can be done using Gauss formula and the data in [Vid05b]. The values are, respectively,

$$\frac{\Gamma(2/3)\Gamma(13/12)}{\Gamma(4/3)\Gamma(5/12)} = \frac{\sqrt{3}+1}{4}, \quad \frac{\Gamma(2/3)\Gamma(7/12)}{4\Gamma(4/3)\Gamma(11/12)} = \frac{3(\sqrt{3}-1)}{4}.$$

The right identification is achieved after conjugating $\sqrt{3}$; this twist is probably preserved under contiguous relations and can be analyzed for each Schwartz type.

Example 4.2 Here we compute a degree 5 Klein's pull-back covering for the local exponent differences $(1/2, 2/3, 2/3)$. Up to a constant multiple, the Schwartz map in Step 4 is

$${}_2F_1\left(\begin{matrix} \frac{1}{4}, -\frac{5}{12} \\ \frac{1}{3} \end{matrix} \middle| X\right) \bigg/ X^{2/3} {}_2F_1\left(\begin{matrix} \frac{1}{4}, \frac{11}{12} \\ \frac{5}{3} \end{matrix} \middle| X\right), \quad (17)$$

The two functions are contiguous to the data base evaluations (42)–(45), though the first function is in the contiguous orbit of the later two evaluations (44)–(45), and vice versa. The numerator function is actually presented in (45). For the denominator ${}_2F_1$ function we have this contiguous relation with $(a, b, c) = (1/4, -1/12, 2/3)$:

$$b(a-c) {}_2F_1\left(\begin{matrix} a, b+1 \\ c+1 \end{matrix} \middle| X\right) = c(a-b) {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix} \middle| X\right) + a(b-c) {}_2F_1\left(\begin{matrix} a+1, b \\ c+1 \end{matrix} \middle| X\right).$$

Using data base formulas (42)–(43), Step 5 returns

$${}_2F_1\left(\begin{matrix} \frac{1}{4}, \frac{11}{12} \\ \frac{5}{3} \end{matrix} \middle| X\right) = \frac{1-x/5}{1+x/4} (1-2x)^{3/4}.$$

In terms of x , the Schwarz map $f_1(X)$ in (17) is a constant times $x^{-2/3}(1+5x/2)/(1-x/5)$. Therefore $z = w x^2(x-5)^3/(5x+2)^3$, where the constant $w = -2$ is obtained by evaluation at $(x, z) = (-4, -4)$. The final step returns $Z = -X^2(4X-5)^3/(5X-4)^3$; a modern computer algebra system has little problem eliminating x, z .

The following transformation of Gauss hypergeometric functions follows:

$${}_2F_1\left(\begin{matrix} \frac{1}{4}, -\frac{5}{12} \\ \frac{1}{3} \end{matrix} \middle| X\right) = \left(1 - \frac{5X}{4}\right)^{1/4} {}_2F_1\left(\begin{matrix} \frac{1}{4}, -\frac{1}{12} \\ \frac{2}{3} \end{matrix} \middle| -\frac{X^2(4X-5)^3}{(5X-4)^3}\right). \quad (18)$$

Example 4.3 Now we compute a degree 7 Klein's pull-back covering for the local exponent differences $(3/2, 1/3, 1/3)$. Up to a constant multiple, the Schwartz map in Step 4 is

$${}_2F_1\left(\begin{matrix} -\frac{1}{4}, -\frac{7}{12} \\ \frac{2}{3} \end{matrix} \middle| X\right) \bigg/ X^{1/3} {}_2F_1\left(\begin{matrix} -\frac{1}{4}, \frac{1}{12} \\ \frac{4}{3} \end{matrix} \middle| X\right), \quad (19)$$

Like in Example 4.1, we have contiguous relations for Euler transformations of these functions. The contiguous relations have to be applied, with $(a, b, c) = (1/4, -1/12, 2/3)$ or

$(1/4, 7/12, 4/3)$, respectively:

$$\begin{aligned}
bc(1-X)^2 {}_2F_1\left(\begin{matrix} a+1, b+1 \\ c \end{matrix} \middle| X\right) &= c((a-c+1)X+b) {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix} \middle| X\right) \\
&\quad + (a+b+1-c)(b-c)X {}_2F_1\left(\begin{matrix} a+1, b \\ c+1 \end{matrix} \middle| X\right), \\
ab(1-X)^2 {}_2F_1\left(\begin{matrix} a+1, b+1 \\ c \end{matrix} \middle| X\right) &= (a-c+1)(aX+b-c+1) {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix} \middle| X\right) \\
&\quad + (a+b+1-c)(c-1) {}_2F_1\left(\begin{matrix} a, b-1 \\ c-1 \end{matrix} \middle| X\right).
\end{aligned}$$

Using data base formulas (42)–(45), Step 5 returns

$$\begin{aligned}
{}_2F_1\left(\begin{matrix} -\frac{1}{4}, -\frac{7}{12} \\ \frac{2}{3} \end{matrix} \middle| \frac{x(x+4)^3}{4(2x-1)^3}\right) &= (1-7x-\frac{7}{2}x^2)(1-2x)^{-7/4}, \\
{}_2F_1\left(\begin{matrix} -\frac{1}{4}, \frac{1}{12} \\ \frac{4}{3} \end{matrix} \middle| \frac{x(x+4)^3}{4(2x-1)^3}\right) &= \frac{1-x-\frac{1}{14}x^2}{1+\frac{1}{4}x}(1-2x)^{-3/4}.
\end{aligned}$$

In Step 6, we conclude that $z = wx(x^2 + 14x - 14)^3 / (7x^2 + 14x - 2)^3$, where the constant $w = -1$ by evaluation $(x, y) = (-4, -4)$. The final step gives

$$Z = -\frac{X(X^2 - 42X - 7)^3}{(7X^2 + 42X - 1)^3}. \quad (20)$$

Similarly to (18), the following transformation of Gauss hypergeometric functions follows:

$${}_2F_1\left(\begin{matrix} -\frac{1}{4}, -\frac{7}{12} \\ \frac{2}{3} \end{matrix} \middle| X\right) = (1 - 42X - 7X^2)^{1/4} {}_2F_1\left(\begin{matrix} \frac{1}{4}, -\frac{1}{12} \\ \frac{2}{3} \end{matrix} \middle| -\frac{X(X^2 - 42X - 7)^3}{(7X^2 + 42X - 1)^3}\right). \quad (21)$$

Here are a few more Klein's pull-back coverings for hypergeometric equations of Schwarz type $(1/2, 1/3, 1/3)$, of degree 7 or 9. The given are local exponent differences and Klein's pull-back coverings:

$$\begin{aligned}
(1/2, 1/3, 4/3): \quad Z &= -\frac{X(256X^2 - 448X + 189)^3}{27(28X - 27)^3}, \\
(1/2, 2/3, 4/3): \quad Z &= \frac{19683X^2(4X - 1)^3}{(256X^3 - 192X^2 + 21X - 4)^3}, \\
(1/2, 1/3, 5/3): \quad Z &= -\frac{19683X(128X - 125)^3}{(16384X^3 - 30720X^2 + 14880X - 625)^3}, \\
(3/2, 1/3, 2/3): \quad Z &= -\frac{729X(5X^2 + 14X + 125)^3}{(4X^3 + 15X^2 - 690X - 625)^3}.
\end{aligned}$$

Next we give Klein's pull-back covering of degree 14 for a hypergeometric equation of the other tetrahedral Schwarz type $(1/3, 1/3, 2/3)$, underscoring the strategy of Step 5 to

keep coefficients of contiguous relations as functions of X only, so to avoid higher degree in the variable x to be eliminated. The simplest Klein's pull-back covering for Schwarz type $(1/3, 1/3, 2/3)$ is a quadratic transformation. Klein's coverings for the local exponent differences $(2/3, 2/3, 2/3)$ or $(1/3, 1/3, 4/3)$ are specializations of classical transformations of degree 6 for Gauss hypergeometric functions. Here are two Klein's coverings of degree 10:

$$\begin{aligned} (1/3, 2/3, 5/3) : \quad Z &= \frac{4X(256X^3 - 640X^2 + 520X - 135)^3}{27(X-1)^2(32X-27)^3}, \\ (2/3, 2/3, 4/3) : \quad Z &= -\frac{X^2(X-1)^2(16X^2 - 16X + 5)^3}{4(5X^2 - 5X + 1)^3}. \end{aligned}$$

Example 4.4 Here we compute Klein's pull-back covering for the local exponent differences $(2/3, 4/3, 4/3)$. We keep the same setting of Step 0, but we have to adjust Step 1. Eventually, the ramification pattern must be $3 + 3 + 3 + 3 + 2 = 4 + 4 + 3 + 3 = 2 + 2 + 2 + 2 + 2 + 2 + 2$, using the notation of [Vid04]. We assign the local exponent $4/3$ to the points $X = 0$ and $X = 1$, and the local exponent $2/3$ to the point $X = \infty$. Since the point $X = 0$ lies above $Z = 0$ by assumption, the point $X = 1$ will end up there as well.

In Step 2 the Darboux covering for the standard equation is the same (11), while the Darboux covering for the given equation H_1 is

$$X = \frac{x(x+2)^3}{(2x+1)^3}. \quad (22)$$

Similarly, in Step 3 the Schwarz map $f_0(Z)$ for the standard equation is the same (12), while the Schwarz map $f_1(X)$ in Step 4 is

$${}_2F_1\left(\begin{matrix} -\frac{1}{2}, -\frac{7}{6} \\ -\frac{1}{3} \end{matrix} \middle| X\right) \bigg/ X^{4/3} {}_2F_1\left(\begin{matrix} \frac{1}{6}, \frac{5}{6} \\ \frac{7}{3} \end{matrix} \middle| X\right). \quad (23)$$

Contiguous relations for these functions are the following, with $(a, b, c) = (1/2, -1/6, 2/3)$ or $(1/6, 5/6, 4/3)$, respectively.

$$\begin{aligned} (1-c) {}_2F_1\left(\begin{matrix} a-1, b-1 \\ c-1 \end{matrix} \middle| X\right) &= ((a-1)X + b - c + 1) {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix} \middle| X\right) + b(1-X) {}_2F_1\left(\begin{matrix} a, b+1 \\ c \end{matrix} \middle| X\right), \\ (c-a)(c-b)X {}_2F_1\left(\begin{matrix} a, b \\ c+1 \end{matrix} \middle| X\right) &= c((c-b)X - c + 1) {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix} \middle| X\right) + c(c-1) {}_2F_1\left(\begin{matrix} a, b-1 \\ c-1 \end{matrix} \middle| X\right), \end{aligned}$$

Utilizing data base formulas (46)–(49) we get the expressions in Step 5:

$${}_2F_1\left(\begin{matrix} -\frac{1}{2}, -\frac{7}{6} \\ -\frac{1}{3} \end{matrix} \middle| X\right) = \left(\frac{1-3X}{2} + \frac{1-X}{2} \frac{(1+2x)^2}{(1-x)^2}\right) \frac{1}{(1+2x)^{1/2}}. \quad (24)$$

$${}_2F_1\left(\begin{matrix} \frac{1}{6}, \frac{5}{6} \\ \frac{7}{3} \end{matrix} \middle| X\right) = \frac{8}{21} \left(\frac{3X-2}{X} + \frac{2}{X} \frac{1+\frac{1}{2}x}{1+2x}\right) \frac{(1+2x)^{1/2} (1+x)^{1/3}}{1+\frac{1}{2}x}. \quad (25)$$

In Step 6 we compute Schwarz map (23):

$$f_1(X, x) = \frac{21(x+2)}{32X^{1/3}(1+x)^{1/3}(1-x)^2} \frac{(1-3X)(1-x)^2 + (1-X)(1+2x)^2}{(3X-2)(1+2x) + (x+2)}. \quad (26)$$

Further evaluation using (22) would probably give a rather nice expression in x only, but we wish to avoid higher degree in x . We may only wish to simplify the power $X^{1/3}$ in the denominator. The Schwarz map $f_1(X, x)$ has to be identified with $f_1(Z)$; we conclude that

$$\frac{1}{z} = \frac{w (2x+1)^3}{x(1+x)(1-x)^6} \left(\frac{(1-3X)(1-x)^2 + (1-X)(1+2x)^2}{(3X-2)(1+2x) + (x+2)} \right)^3 \quad (27)$$

for some constant w . To find the constant, we evaluate at $z = -4$, $x = -2$ and $X = 0$; this is compatible with (22), and there are no zeroes in multiplicative terms. We find $w = 1/2$.

In Step 7 we can use straightforward means (within a modern computer algebra system like **Maple**) to eliminate x and z , and get the following unique result:

$$Z = -\frac{108 X^4 (X-1)^4 (27X^2 - 27X + 7)^3}{(189X^4 - 378X^3 + 301X^2 - 112X + 16)^3}. \quad (28)$$

5 Two icosahedral examples

Computation of Klein's pull-back coverings for icosahedral hypergeometric equations is a level harder than for tetrahedral or octahedral hypergeometric equations. As Darboux coverings have degree 12 and Darboux curves can be non-rational, straightforward elimination methods are inadequate. The following two examples demonstrate some of simplification techniques suggested at the end of Section 3.

The standard hypergeometric equation for the icosahedral monodromy group has the local exponent differences $(1/2, 1/3, 1/5)$. We assign these local exponents to, respectively, $Z = 1$, $Z = \infty$ and $Z = 0$. The Darboux covering $Z = \phi_0(z)$ in Step 2 is given in (58):

$$Z = \frac{1728 z (z^2 - 11z - 1)^5}{(z^4 + 228z^3 + 494z^2 - 228z + 1)^3}. \quad (29)$$

The Schwarz map in Step 3 is

$$f_0(Z) = {}_2F_1\left(\begin{matrix} \frac{19}{60}, -\frac{1}{60} \\ \frac{4}{5} \end{matrix} \middle| Z\right) \bigg/ Z^{1/5} {}_2F_1\left(\begin{matrix} \frac{11}{60}, \frac{59}{60} \\ \frac{6}{5} \end{matrix} \middle| Z\right) = \left(-\frac{1}{1728 z}\right)^{1/5}, \quad (30)$$

evaluated using (29) and (59), (61).

Example 5.1 Here we compute a degree 11 pull-back covering for hypergeometric equations with the local exponent differences $(1/2, 2/3, 1/5)$. In Step 1, we choose the points $X = 0$, $X = 1$, $X = \infty$ to lie above the points $Z = 0$, $Z = 1$, $Z = \infty$, respectively. The Darboux covering $X = \phi_1(x)$ in Step 2 mimics (29):

$$X = \frac{1728 x (x^2 - 11x - 1)^5}{(x^4 + 228x^3 + 494x^2 - 228x + 1)^3}. \quad (31)$$

The Schwartz map in Step 4 is

$${}_2F_1\left(\begin{matrix} \frac{29}{60}, -\frac{11}{60} \\ \frac{4}{5} \end{matrix} \middle| X\right) \bigg/ X^{1/5} {}_2F_1\left(\begin{matrix} \frac{1}{60}, \frac{41}{60} \\ \frac{6}{5} \end{matrix} \middle| X\right), \quad (32)$$

An explicit expression for Euler's transformation of the numerator function is presented directly in (60), while Euler's transformation for the denominator function can be computed from contiguous relation (16) with $(a, b, c) = (11/60, 31/60, 6/5)$ and data base formulas (61)–(62). Step 5 returns the explicit expressions

$$\begin{aligned} {}_2F_1\left(-\frac{11}{60}, \frac{29}{60} \middle| X\right) &= (1 + 66x - 11x^2)(1 - 228x + 494x^2 + 228x^3 + x^4)^{-11/20}, \\ {}_2F_1\left(\frac{1}{60}, \frac{41}{60} \middle| X\right) &= \frac{1 + 6x - \frac{1}{11}x^2}{1 + 11x - x^2}(1 - 228x + 494x^2 + 228x^3 + x^4)^{1/20}. \end{aligned}$$

Up to a constant multiple, we have $z = wx(11 + 66x - x^2)^5/(1 + 66x - 11x^2)^5$ in Step 6. The constant $w = -1$ is determined by the evaluation $x = z = (11 + 5\sqrt{5})/2$. The alternative evaluation $z = (11 - 5\sqrt{5})/2$ gives $w = (123 - 55\sqrt{5})/2$, but then the relation between x and z is not invariant under the given automorphisms $x = -1/x$, $z \mapsto -1/z$.

To simplify equations (29), (31) and the relation for z with $w = -1$, we introduce the variables $\alpha = x - 1/x$ and $\beta = z - 1/z$. Due to the symmetry between higher and lower coefficients in the polynomial factors in (29) and (31), we can “replace” x, z by the variables α, β ; we get the simpler expressions

$$X = \frac{1728(\alpha - 11)^5}{(\alpha^2 + 228\alpha + 496)^3}, \quad Z = \frac{1728(\beta - 11)^5}{(\beta^2 + 228\beta + 496)^3}. \quad (33)$$

The expression for z with $w = -1$ is replaced by

$$\beta = z - \frac{1}{z} = -\frac{\alpha^{11} - 660\alpha^{10} - 25937228680\alpha^9 + \dots}{(11\alpha^2 - 792\alpha + 4256)^5}.$$

Elimination of α, β is rather straightforward now. The result is

$$Z = \frac{X(102400X^2 - 11264X - 11)^5}{(180224000X^3 + 4325376X^2 - 21252X + 1)^3}. \quad (34)$$

Example 5.2 Here we compute a degree 18 pull-back covering for hypergeometric equations with the local exponent differences $(1/5, 1/5, 6/5)$. The Darboux curve is of genus 1, defined in (97). In Step 1, we copy the Darboux covering from (98):

$$X = \frac{16\xi(1+x-x^2)^2(1-\xi)^2}{(1+\xi+2x)(1+\xi-2x)^5}, \quad \xi^2 = x(1+x-x^2). \quad (35)$$

In Step 3, the Schwartz map $f_1(X)$ is

$${}_2F_1\left(\frac{9}{10}, -\frac{3}{10} \middle| X\right) \bigg/ X^{1/5} {}_2F_1\left(\frac{11}{10}, -\frac{1}{10} \middle| X\right), \quad (36)$$

Contiguous relations for these functions are the following, with $(a, b, c) = (7/10, -1/10, 4/5)$ or $(1/10, 9/10, 6/5)$, respectively:

$$\begin{aligned} {}_c {}_2F_1\left(\begin{matrix} a-1, b+1 \\ c \end{matrix} \middle| X\right) &= {}_c {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix} \middle| X\right) + (a-b-1)X {}_2F_1\left(\begin{matrix} a, b+1 \\ c+1 \end{matrix} \middle| X\right), \\ a(b-c) {}_2F_1\left(\begin{matrix} a+1, b-1 \\ c \end{matrix} \middle| X\right) &= (b-1)(a-c+1) {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix} \middle| X\right) + (c-1)(b-a-1) {}_2F_1\left(\begin{matrix} a, b-1 \\ c-1 \end{matrix} \middle| X\right). \end{aligned}$$

Utilizing data base formulas (99)–(102) we get these expressions in Step 5:

$${}_2F_1\left(\begin{matrix} \frac{9}{10}, -\frac{3}{10} \\ \frac{4}{5} \end{matrix} \middle| X\right) = \frac{(1-2x-3y)(1-\xi+2x)^{1/15}}{(1-\xi)^{7/5}(1+\xi+2x)^{7/30}(1+\xi-2x)^{3/2}}, \quad (37)$$

$${}_2F_1\left(\begin{matrix} \frac{11}{10}, -\frac{1}{10} \\ \frac{6}{5} \end{matrix} \middle| X\right) = \frac{(3\xi-2x-x^2)(1+\xi)^{1/10}(1-\xi)^{3/10}}{3\xi(1-\xi+2x)^{1/30}(1+\xi+2x)^{2/15}\sqrt{1+\xi-2x}}. \quad (38)$$

Step 6 gives a Schwarz map $f_1(x, \xi)$ and the following relation:

$$z = \frac{w(1-x)(3\xi-2x-x^2)^5}{x^2(1+x)(3\xi+2x-1)^5}, \quad (39)$$

where $w = 1$ by evaluation at $(x, \xi) = ((1+\sqrt{5})/2, 0)$ and $z = (11+5\sqrt{5})/2$. We use the compatible automorphisms $(x, \xi) \mapsto (-1/x, \xi/x^2)$ and $z \mapsto -1/z$ of Darboux coverings to check the value of w , and simplify equations for Step 7. Correspondingly, we introduce the new variables

$$\alpha = x - \frac{1}{x}, \quad \zeta = \frac{\xi(x^2+1)}{x^2}, \quad \beta = z - \frac{1}{z}.$$

Equations (35) can be simplified to

$$\frac{1}{X} = \frac{1}{2} - \frac{\zeta(\alpha^2 - 22\alpha - 4)}{16(\alpha - 1)^3}, \quad \zeta^2 = (1 - \alpha)(4 + \alpha^2).$$

From here we can eliminate ζ linearly and get the following equation of degree 6 for α :

$$\alpha(\alpha+4)^5 X^2 - 256(\alpha-1)^5 X + 256(\alpha-1)^5 = 0 \quad (40)$$

Alternatively, one can express all cubic monomials in x, ξ in lower degree monomials by a Gröbner basis. That would be a basis to express

$$\beta = z - \frac{1}{z} = \frac{-6\zeta(5\alpha^7 + 65150\alpha^6 + \dots) + (\alpha^9 + 58662\alpha^8 + \dots)}{\alpha(3\zeta + 11\alpha - 6)^5}$$

and a few more (a priori specified) increasingly invariant expressions towards Z ; we would keep at most 6 low degree monomials in α, ζ . But automatic computation from this point with Gröbner basis in all variables does not look feasible. Without a resolvent sequence, a practical trick is to take (the numerators of) the factors $\beta - 11$ and $\beta^2 + 228\beta + 496$ in (33), “eliminate” α, ζ from them using resultants, and recognize suitable polynomial factors in X

for a possible final expression, keeping in mind its whole ramification pattern. Candidate expressions can be normalized by the condition that other ramification points must lie above $Z = 1$. In this case, we get a unique candidate factor $512X^2 - 512X + 3$ for the numerator, and a unique factor in X for the denominator, and an additional ramification point is $X = 1/2$. The final result is

$$Z = -\frac{108 X (X - 1) (512X^2 - 512X + 3)^5}{(1048576X^6 - 3145728X^5 + 3244032X^4 - 1245184X^3 + 94848X^2 + 3456X + 1)^3}. \quad (41)$$

6 A data basis of hypergeometric identities

Here we present a data base of Darboux coverings and Darboux evaluations of algebraic hypergeometric solutions with small local exponent differences. For each Schwarz type we evaluate four functions: two hypergeometric solutions of the main representative hypergeometric equation, and a contiguous “companion” function to each of these two functions. This allows evaluation of all other hypergeometric functions of the same Schwarz type by using contiguous relations and Euler’s formula (10). For example, (42) and (44) represent solutions of the hypergeometric equation with local exponent differences $1/2, 1/3, 1/3$ at $z = 1, 0, \infty$, respectively; and (43) and (45) evaluate contiguous functions to (42) and (44), respectively.

The formulas can be checked by expanding power series at $x = 0$ on both sides, also in the cases of genus 1 Darboux curves, as $\xi = O(\sqrt{x})$ for these curves.

6.1 Tetrahedral hypergeometric equations

For the Schwarz type $(1/2, 1/3, 1/3)$ we give the following evaluations. The Darboux covering is evident from the arguments of the ${}_2F_1$ series.

$${}_2F_1\left(\begin{matrix} 1/4, -1/12 \\ 2/3 \end{matrix} \middle| \frac{x(x+4)^3}{4(2x-1)^3}\right) = (1-2x)^{-1/4}. \quad (42)$$

$${}_2F_1\left(\begin{matrix} 5/4, -1/12 \\ 5/3 \end{matrix} \middle| \frac{x(x+4)^3}{4(2x-1)^3}\right) = \frac{1+x}{(1+\frac{1}{4}x)^2} (1-2x)^{-1/4}. \quad (43)$$

$${}_2F_1\left(\begin{matrix} 1/4, 7/12 \\ 4/3 \end{matrix} \middle| \frac{x(x+4)^3}{4(2x-1)^3}\right) = \frac{1}{1+\frac{1}{4}x} (1-2x)^{3/4}. \quad (44)$$

$${}_2F_1\left(\begin{matrix} 1/4, -5/12 \\ 1/3 \end{matrix} \middle| \frac{x(x+4)^3}{4(2x-1)^3}\right) = (1+\frac{5}{2}x) (1-2x)^{-5/4}. \quad (45)$$

For the Schwarz type $(1/3, 1/3, 2/3)$ we give the following evaluations.

$${}_2F_1\left(\begin{matrix} 1/2, -1/6 \\ 2/3 \end{matrix} \middle| \frac{x(x+2)^3}{(2x+1)^3}\right) = (1+2x)^{-1/2}. \quad (46)$$

$${}_2F_1\left(\begin{matrix} 1/2, 5/6 \\ 2/3 \end{matrix} \middle| \frac{x(x+2)^3}{(2x+1)^3}\right) = \frac{1}{(1-x)^2} (1+2x)^{3/2}. \quad (47)$$

$${}_2F_1\left(\begin{matrix} 1/6, 5/6 \\ 4/3 \end{matrix} \middle| \frac{x(x+2)^3}{(2x+1)^3}\right) = \frac{1}{1+\frac{1}{2}x} (1+2x)^{1/2} (1+x)^{1/3}. \quad (48)$$

$${}_2F_1\left(\begin{matrix} 1/6, -1/6 \\ 1/3 \end{matrix} \middle| \frac{x(x+2)^3}{(2x+1)^3}\right) = (1+2x)^{-1/2} (1+x)^{1/3}. \quad (49)$$

6.2 Octahedral hypergeometric equations

For the Swartz type $(1/2, 1/3, 1/4)$ we have the following evaluations.

$${}_2F_1\left(\begin{matrix} 7/24, -1/24 \\ 3/4 \end{matrix} \middle| \frac{108x(x-1)^4}{(x^2+14x+1)^3}\right) = (1+14x+x^2)^{-1/8}. \quad (50)$$

$${}_2F_1\left(\begin{matrix} 7/24, 23/24 \\ 7/4 \end{matrix} \middle| \frac{108x(x-1)^4}{(x^2+14x+1)^3}\right) = \frac{1+2x-\frac{1}{11}x^2}{(1-x)^3} (1+14x+x^2)^{7/8}. \quad (51)$$

$${}_2F_1\left(\begin{matrix} 5/24, 13/24 \\ 5/4 \end{matrix} \middle| \frac{108x(x-1)^4}{(x^2+14x+1)^3}\right) = \frac{1}{1-x} (1+14x+x^2)^{5/8}. \quad (52)$$

$${}_2F_1\left(\begin{matrix} 5/24, -11/24 \\ 1/4 \end{matrix} \middle| \frac{108x(x-1)^4}{(x^2+14x+1)^3}\right) = \frac{1-22x-11x^2}{(1+14x+x^2)^{11/8}}. \quad (53)$$

For the Swartz type $(1/4, 1/4, 2/3)$ we have the following evaluations.

$${}_2F_1\left(\begin{matrix} 7/12, -1/12 \\ 3/4 \end{matrix} \middle| \frac{27x(x+1)^4}{2(x^2+4x+1)^3}\right) = \frac{(1+\frac{1}{2}x)^{1/4}}{(1+4x+x^2)^{1/4}}. \quad (54)$$

$${}_2F_1\left(\begin{matrix} 7/12, 11/12 \\ 7/4 \end{matrix} \middle| \frac{27x(x+1)^4}{2(x^2+4x+1)^3}\right) = \frac{(1+\frac{1}{2}x)^{1/4} (1+4x+x^2)^{7/4}}{(1+x)^3}. \quad (55)$$

$${}_2F_1\left(\begin{matrix} 1/6, 5/6 \\ 5/4 \end{matrix} \middle| \frac{27x(x+1)^4}{2(x^2+4x+1)^3}\right) = \frac{(1+2x)^{1/4} (1+4x+x^2)^{1/2}}{1+x}. \quad (56)$$

$${}_2F_1\left(\begin{matrix} 1/6, -1/6 \\ 1/4 \end{matrix} \middle| \frac{27x(x+1)^4}{2(x^2+4x+1)^3}\right) = \frac{(1+2x)^{1/4}}{(1+4x+x^2)^{1/2}}. \quad (57)$$

6.3 Icosahedral hypergeometric equations

The Darboux covering for hypergeometric equations of the Schwarz type $(1/2, 1/3, 1/5)$ is the following:

$$\varphi_1(x) = \frac{1728x(x^2-11x-1)^5}{(x^4+228x^3+494x^2-228x+1)^3} \quad (58)$$

The simplest evaluations for this Schwarz type are these:

$${}_2F_1\left(\begin{matrix} 19/60, -1/60 \\ 4/5 \end{matrix} \middle| \varphi_1(x)\right) = (1-228x+494x^2+228x^3+x^4)^{-1/20}. \quad (59)$$

$${}_2F_1\left(\begin{matrix} 19/60, 59/60 \\ 4/5 \end{matrix} \middle| \varphi_1(x)\right) = \frac{(1+66x-11x^2)(1-228x+494x^2+228x^3+x^4)^{19/20}}{(1+x^2)(1+522x-10006x^2-522x^3+x^4)}. \quad (60)$$

$${}_2F_1\left(\begin{matrix} 11/60, 31/60 \\ 6/5 \end{matrix} \middle| \varphi_1(x)\right) = \frac{(1-228x+494x^2+228x^3+x^4)^{11/20}}{1+11x-x^2}. \quad (61)$$

$${}_2F_1\left(\begin{matrix} 11/60, -29/60 \\ 1/5 \end{matrix} \middle| \varphi_1(x)\right) = \frac{1+435x-6670x^2-3335x^4-87x^5}{(1-228x+494x^2+228x^3+x^4)^{29/20}}. \quad (62)$$

The Darboux covering for hypergeometric equations of the Schwarz type $(1/2, 1/3, 2/5)$ is the same $\varphi_1(x)$. Here are the simplest contiguous evaluations:

$${}_2F_1\left(\begin{matrix} 13/60, -7/60 \\ 3/5 \end{matrix} \middle| \varphi_1(x)\right) = \frac{1-7x}{(1-228x+494x^2+228x^3+x^4)^{7/20}}. \quad (63)$$

$${}_2F_1\left(\begin{matrix} 13/60, 53/60 \\ 3/5 \end{matrix} \middle| \varphi_1(x)\right) = \frac{(1+119x+187x^2+17x^3)(1-228x+494x^2+228x^3+x^4)^{13/20}}{(1+x^2)(1+522x-10006x^2-522x^3+x^4)}.$$

$${}_2F_1\left(\begin{matrix} 17/60, 37/60 \\ 7/5 \end{matrix} \middle| \varphi_1(x)\right) = \frac{(1+\frac{1}{7}x)(1-228x+494x^2+228x^3+x^4)^{17/20}}{(1+11x-x^2)^2}. \quad (64)$$

$${}_2F_1\left(\begin{matrix} 17/60, -23/60 \\ 2/5 \end{matrix} \middle| \varphi_1(x)\right) = \frac{(1+207x-391x^2+1173x^3+46x^4)}{(1-228x+494x^2+228x^3+x^4)^{23/20}}. \quad (65)$$

The Darboux covering for the Schwarz type $(1/2, 1/5, 2/5)$ is the following:

$$\varphi_2(x) = \frac{64x(x^2-x-1)^5}{(x^2-1)(x^2+4x-1)^5}. \quad (66)$$

The simplest evaluations are the following:

$${}_2F_1\left(\begin{matrix} 7/20, -1/20 \\ 4/5 \end{matrix} \middle| \varphi_2(x)\right) = \frac{(1+x)^{7/20}}{(1-x)^{1/20}(1-4x-x^2)^{1/4}}. \quad (67)$$

$${}_2F_1\left(\begin{matrix} 7/20, 19/20 \\ 4/5 \end{matrix} \middle| \varphi_2(x)\right) = \frac{(1+3x)(1+x)^{7/20}(1-x)^{19/20}(1-4x-x^2)^{7/4}}{(1+x^2)(1+22x-6x^2-22x^3+x^4)}. \quad (68)$$

$${}_2F_1\left(\begin{matrix} 3/20, 11/20 \\ 6/5 \end{matrix} \middle| \varphi_2(x)\right) = \frac{(1+x)^{3/20}(1-x)^{11/20}(1-4x-x^2)^{3/4}}{1+x-x^2}. \quad (69)$$

$${}_2F_1\left(\begin{matrix} 3/20, -9/20 \\ 1/5 \end{matrix} \middle| \varphi_2(x)\right) = \frac{(1+12x-6x^2-2x^3-9x^4)(1+x)^{3/20}}{(1-x)^{9/20}(1-4x-x^2)^{9/4}}. \quad (70)$$

For other icosahedral Schwarz types, the Darboux curves are not rational but have genus 1. We use then two algebraically related variables x, ξ in Algorithm 3.1.

The Darboux curve for hypergeometric equations of the Schwarz type $(1/3, 1/3, 2/5)$ is given by the equation

$$E_3 : \quad \xi^2 = x(1+33x-9x^2). \quad (71)$$

The Darboux covering is

$$\varphi_3(x, \xi) = \frac{144\xi(1+33x-9x^2)^2(1-9\xi+54x)}{(1+21\xi-117x+9x\xi-234x^2)^3} \quad (72)$$

Here are simplest evaluations for the Schwarz type $(1/3, 1/3, 2/5)$.

$${}_2F_1\left(\begin{matrix} 3/10, -1/30 \\ 3/5 \end{matrix} \middle| \varphi_3(x, \xi)\right) = \frac{(1-9\xi+54x)^{1/30}}{(1+21\xi-117x+9x\xi-234x^2)^{1/10}}. \quad (73)$$

$${}_2F_1\left(\begin{matrix} 3/10, 29/30 \\ 3/5 \end{matrix} \middle| \varphi_3(x, \xi)\right) =$$

$$\frac{(1+21\xi-117x+9x\xi-234x^2)^{9/10}(1+9x)^2(1+198x-99x^2)}{(1-9\xi+54x)^{29/30}(1-21\xi-117x-9x\xi-234x^2)^2}. \quad (74)$$

$${}_2F_1\left(\begin{matrix} 7/10, 11/30 \\ 7/5 \end{matrix} \middle| \varphi_3(x, \xi)\right) = \frac{(1+21\xi-117x+9x\xi-234x^2)^{11/10}}{(1-9\xi+54x)^{11/30}(1+33x-9x^2)}. \quad (75)$$

$${}_2F_1\left(\begin{matrix} -3/10, 11/30 \\ 2/5 \end{matrix} \middle| \varphi_3(x, \xi)\right) = \frac{(1-9\xi+54x)^{19/30}(1-15\xi-72x-54x^2)}{(1+21\xi-117x+9x\xi-234x^2)^{9/10}(1+9x)}. \quad (76)$$

The Darboux curve and covering for the Schwarz type $(1/3, 2/3, 1/5)$ are the same as in (71) and (72). The simplest evaluations are:

$${}_2F_1\left(\begin{matrix} -1/10, 17/30 \\ 4/5 \end{matrix} \middle| \varphi_3(x, \xi)\right) = \frac{(1-9\xi+54x)^{13/30}}{(1+21\xi-117x+9x\xi-234x^2)^{3/10}}. \quad (77)$$

$${}_2F_1\left(\begin{matrix} 9/10, 17/30 \\ 9/5 \end{matrix} \middle| \varphi_3(x, \xi)\right) = \frac{(1+21\xi-117x+9x\xi-234x^2)^{17/10}(1+\frac{3}{7}x)}{(1-9\xi+54x)^{17/30}(1+33x-9x^2)^2}. \quad (78)$$

$${}_2F_1\left(\begin{matrix} 1/10, 23/30 \\ 6/5 \end{matrix} \middle| \varphi_3(x, \xi)\right) = \frac{(1+21\xi-117x+9x\xi-234x^2)^{3/10}(1-9\xi+54x)^{7/30}(\xi+5x)}{\xi(1+9x)}. \quad (79)$$

$${}_2F_1\left(\begin{matrix} 1/10, -7/30 \\ 1/5 \end{matrix} \middle| \varphi_3(x, \xi)\right) = \frac{(1-9\xi+54x)^{7/30}(1-21x)}{(1+21\xi-117x+9x\xi-234x^2)^{7/10}}. \quad (80)$$

The Darboux curve for the Schwarz type $(2/3, 1/5, 1/5)$ can be given by

$$E_4: \quad \xi^2 = x(1+5x-5x^2). \quad (81)$$

The Darboux covering is given by

$$\varphi_4(x, \xi) = \frac{432x(1-\frac{7}{5}\xi-9x-x^2)^5(1+50x-125\xi^2+450x\xi-500x^2)}{(5\xi+57x)(1+\frac{18}{5}\xi-16x+x^2)^5(1+50x-125\xi^2-450x\xi-500x^2)}. \quad (82)$$

The simplest evaluations are:

$${}_2F_1\left(\begin{matrix} 1/6, -1/30 \\ 4/5 \end{matrix} \middle| \varphi_4(x, \xi)\right) = \frac{(1-\frac{3}{5}\xi-\frac{34}{5}x)^{1/6}}{(1+3\xi-20x)^{1/6}(1+50x-125\xi^2-450x\xi-500x^2)^{1/30}}. \quad (83)$$

$${}_2F_1\left(\begin{matrix} 1/6, 29/30 \\ 4/5 \end{matrix} \middle| \varphi_4(x, \xi)\right) = \frac{(1+3\xi-20x)^{5/6}(1-\frac{3}{5}\xi-\frac{34}{5}x)^{1/6}(1-\frac{35}{4}\xi-\frac{101}{4}x)}{(1+50x-125\xi^2-450x\xi-500x^2)^{1/30}(1-\frac{95}{4}\xi+\frac{83}{4}x+\frac{21}{4}\xi^2+\frac{475}{4}x\xi+10x^2)}. \quad (84)$$

$${}_2F_1\left(\begin{matrix} 1/6, 11/30 \\ 6/5 \end{matrix} \middle| \varphi_4(x, \xi)\right) = (1+50x-125\xi^2-450x\xi-500x^2)^{11/30} \times \frac{(1+3\xi-20x)^{5/6}(1-\frac{3}{5}\xi-\frac{34}{5}x)^{1/6}(1+\frac{21}{4}\xi+\frac{41}{4}x)}{(1-9x)(1-\frac{7}{4}\xi-\frac{15}{2}x)(1+5\xi+10x)}. \quad (85)$$

$${}_2F_1\left(\begin{matrix} 1/6, 11/30 \\ 1/5 \end{matrix} \middle| \varphi_4(x, \xi)\right) = (1+50x-125\xi^2-450x\xi-500x^2)^{11/30} \times \frac{(1+3\xi-20x)^{5/6} (1-\frac{3}{5}\xi-\frac{34}{5}x)^{1/6} (1+\frac{21}{4}\xi+\frac{41}{4}x)}{(1-\frac{95}{4}\xi+\frac{83}{4}x+\frac{21}{4}\xi^2+\frac{475}{4}x\xi+10x^2) (1+5\xi+10x)}. \quad (86)$$

The Darboux curve and covering for the Schwarz type $(1/3, 2/5, 3/5)$ are the same as in (81) and (82). The simplest evaluations are:

$${}_2F_1\left(\begin{matrix} -1/6, 13/30 \\ 3/5 \end{matrix} \middle| \varphi_4(x, \xi)\right) = \frac{(1+50x-125\xi^2-450x\xi-500x^2)^{13/30} (1-3\xi+2x)}{(1+3\xi-20x)^{5/6} (1-\frac{3}{5}\xi-\frac{34}{5}x)^{1/6} (1+5\xi+10x)}. \quad (87)$$

$${}_2F_1\left(\begin{matrix} 5/6, 13/30 \\ 3/5 \end{matrix} \middle| \varphi_4(x, \xi)\right) = (1+50x-125\xi^2-450x\xi-500x^2)^{13/30} \times \frac{(1+3\xi-20x)^{13/6} (1-\frac{7}{4}\xi+\frac{25}{2}x-\frac{245}{4}x^2) (1-\frac{7}{20}\xi-\frac{79}{20}x)}{(1-\frac{3}{5}\xi-\frac{34}{5}x)^{1/6} (1-\frac{95}{4}\xi+\frac{83}{4}x+\frac{21}{4}\xi^2+\frac{475}{4}x\xi+10x^2)^2 (1-5x)^2} \quad (88)$$

$${}_2F_1\left(\begin{matrix} 5/6, 7/30 \\ 7/5 \end{matrix} \middle| \varphi_4(x, \xi)\right) = (1+50x-125\xi^2-450x\xi-500x^2)^{7/30} \times \frac{(1+\frac{18}{5}\xi-16x+x^2)^{7/6} (1+\frac{1}{25}x)^{5/6} (1+5\xi+10x)}{(1-\frac{7}{5}\xi-9x-x^2)^2 (1-5x)^{7/6}}. \quad (89)$$

$${}_2F_1\left(\begin{matrix} -1/6, 7/30 \\ 2/5 \end{matrix} \middle| \varphi_4(x, \xi)\right) = \frac{(1+50x-125\xi^2-450x\xi-500x^2)^{7/30} (1-\frac{27}{5}\xi+\frac{58}{5}x-2x^2)}{(1+\frac{18}{5}\xi-16x+x^2)^{5/6} (1+\frac{1}{25}x)^{1/6} (1-5x)^{7/6}} \quad (90)$$

The Darboux curve for the Schwarz type $(1/3, 1/5, 3/5)$ can be given by

$$E_5 : \quad \xi^2 = x(1+x)(1+16x). \quad (91)$$

The Darboux covering can be given by

$$\varphi_5(x, \xi) = -\frac{54(\xi+5x)^3(1-2\xi+6x)^5}{(1-16x^2)(\xi-5x)^2(1-2\xi-14x)^5}. \quad (92)$$

$${}_2F_1\left(\begin{matrix} -1/15, 8/15 \\ 4/5 \end{matrix} \middle| \varphi_5(x, \xi)\right) = \frac{(1+4x)^{8/15} (\xi+5x)^{1/6} x^{1/15}}{(1-2\xi-14x)^{1/3} (\xi-3x)^{3/10}}. \quad (93)$$

$${}_2F_1\left(\begin{matrix} 14/15, 8/15 \\ 9/5 \end{matrix} \middle| \varphi_5(x, \xi)\right) = \frac{(1-2\xi-14x)^{8/3} (1+\frac{2}{3}\xi+\frac{2}{3}x-\frac{16}{3}x^2) (1+4x)^{8/15} (\xi-5x)^2 x^{1/15}}{(1-2\xi+6x)^4 (\xi+5x)^{11/6} (\xi-3x)^{3/10}}. \quad (94)$$

$${}_2F_1\left(\begin{matrix} 2/15, 11/15 \\ 6/5 \end{matrix} \middle| \varphi_5(x, \xi)\right) = \frac{(1-\xi+x)(1-2\xi-14x)^{2/3} (\xi+5x)^{1/6} (\xi-3x)^{13/10}}{(1+\xi+x)(1-2\xi+6x)(1+4x)^{13/15} x^{11/15}}. \quad (95)$$

$${}_2F_1\left(\begin{matrix} 2/15, -4/15 \\ 1/5 \end{matrix} \middle| \varphi_5(x, \xi)\right) = \frac{(1+3\xi+x)(1+4x)^{2/15} (\xi+5x)^{7/6} (\xi-3x)^{3/10}}{(1+\xi+x)(1-2\xi-14x)^{4/3} x^{11/15}}. \quad (96)$$

The Darboux curve for the Schwarz type $(1/5, 1/5, 4/5)$ can be given by

$$E_6 : \quad \xi^2 = x(1+x-x^2). \quad (97)$$

The Darboux covering is given by

$$\varphi_6(x, \xi) = \frac{16 \xi (1+x-x^2)^2 (1-\xi)^2}{(1+\xi+2x)(1+\xi-2x)^5}. \quad (98)$$

The simplest evaluations are:

$${}_2F_1\left(\begin{matrix} 7/10, -1/10 \\ 4/5 \end{matrix} \middle| \varphi_6(x, \xi)\right) = \frac{(1-\xi+2x)^{1/15} (1-\xi)^{3/5}}{(1+\xi+2x)^{7/30} \sqrt{1+\xi-2x}} \quad (99)$$

$${}_2F_1\left(\begin{matrix} 7/10, 9/10 \\ 9/5 \end{matrix} \middle| \varphi_6(x, \xi)\right) = \frac{(1-\xi+2x)^{1/15} (1+\xi+2x)^{23/30} (1+\xi-2x)^{7/2}}{(1-\xi)^{7/5} (1+x-x^2)^2} \quad (100)$$

$${}_2F_1\left(\begin{matrix} 1/10, 9/10 \\ 6/5 \end{matrix} \middle| \varphi_6(x, \xi)\right) = \frac{(\xi+2x+x^2) (1+\xi)^{1/10} (1-\xi)^{3/10}}{\xi (1-\xi+2x)^{1/30} (1+\xi+2x)^{2/15} \sqrt{1+\xi-2x}}. \quad (101)$$

$${}_2F_1\left(\begin{matrix} 1/10, -1/10 \\ 1/5 \end{matrix} \middle| \varphi_6(x, \xi)\right) = \frac{(1+\xi)^{1/10} (1-\xi)^{3/10}}{(1-\xi+2x)^{1/30} (1+\xi+2x)^{2/15} \sqrt{1+\xi-2x}}. \quad (102)$$

The Darboux curve and covering for the Schwarz type $(2/5, 2/5, 2/5)$ are the same as in (97) and (98). The simplest evaluations are:

$${}_2F_1\left(\begin{matrix} 3/10, -1/10 \\ 3/5 \end{matrix} \middle| \varphi_6(x, \xi)\right) = \frac{(1-\xi+2x)^{2/15} (1+\xi+2x)^{1/30} (1-\xi)^{1/5}}{\sqrt{1+\xi-2x}}. \quad (103)$$

$${}_2F_1\left(\begin{matrix} 3/10, 9/10 \\ 8/5 \end{matrix} \middle| \varphi_6(x, \xi)\right) = \frac{(1-\xi+2x)^{2/15} (1+\xi+2x)^{1/30} (1-\xi)^{1/5} (1+\xi-2x)^{3/2} (1+\frac{1}{2}\xi+\frac{1}{2}x)}{(1+x-x^2) (1-\xi+x-x^2)}. \quad (104)$$

$${}_2F_1\left(\begin{matrix} 3/10, 7/10 \\ 7/5 \end{matrix} \middle| \varphi_6(x, \xi)\right) = \frac{(1+\xi+2x)^{7/30} (1+\xi)^{1/5} (1+\xi-2x)^{3/2}}{(1-\xi+2x)^{1/15} (1-\xi)^{2/5} (1+x-x^2)}. \quad (105)$$

$${}_2F_1\left(\begin{matrix} 3/10, -3/10 \\ 2/5 \end{matrix} \middle| \varphi_6(x, \xi)\right) = \frac{(1+\xi+2x)^{7/30} (1+\xi)^{1/5} (1-3\xi+4x-2x^2)}{(1-\xi+2x)^{1/15} (1-\xi)^{2/5} (1+\xi-2x)^{3/2}}. \quad (106)$$

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